ON THE BICANONICAL MORPHISM OF QUADRUPLE GALOIS CANONICAL COVERS

FRANCISCO JAVIER GALLEGO AND BANGERE P. PURNAPRAJNA

Abstract. In this article we study the bicanonical map φ_2 of quadruple Galois canonical covers X of surfaces of minimal degree. We show that φ_2 has diverse behavior and exhibit most of the complexities that are possible for a bicanonical map of surfaces of general type, depending on the type of X. There are cases in which φ_2 is an embedding, and if so happens, φ_2 embeds X as a projectively normal variety, and cases in which φ_2 is not an embedding. If the latter, φ_2 is finite of degree 1, 2 or 4. We also study the canonical ring of X, proving that it is generated in degree less than or equal to 3 and finding the number of generators in each degree. For generators of degree 2 we find a nice general formula which holds for canonical covers of arbitrary degrees. We show that this formula depends only on the geometric and the arithmetic genus of X.

Introduction

Canonical covers of surfaces of minimal degree have a ubiquitous presence in diverse contexts in the geometry of surfaces and threefolds. For example they appear in the classification of surfaces of general type with small c_1^2 as shown in the work of Horikawa, and play an important role in mapping the geography of surfaces of general type. They appear as unavoidable boundary cases in the study of linear series on Calabi–Yau threefolds as the works of Beltrametti and Szemberg (see [BS00]), Oguiso and Peternell (see [OP95]) and the authors (see [GP98]) show, and in the study of the canonical ring of a variety of general type as can be seen in the article of Green [Gre82]. They are also a useful source in constructing new examples of surfaces of general type.

Double and triple canonical covers were classified by Horikawa (see [Hor76]) and Konno (see [Kon91]). In [GP07] and [GP08], the authors classified Galois canonical covers of degree 4. The classification showed that quadruple canonical covers behave quite differently from canonical covers of all other degrees; for instance, quadruple canonical covers are the only covers that admit families with unbounded geometric genus and families with unbounded irregularity. Hence, the geography of Chern numbers of quadruple canonical covers is much more complex.

One of the much studied objects for surfaces of general type is the bicanonical map. In this article we prove results on the bicanonical morphisms of quadruple canonical covers of surfaces of minimal degree. Our results show that the behavior of these bicanonical maps is quite generic, that is, it exhibits all the diversities and complexities that are possible for a bicanonical map of a surface of general type. All of this is amply illustrated in the following theorem:

Theorem 0.1. *Let $\varphi : X \longrightarrow W$ be a quadruple Galois canonical cover of a surface W of minimal degree. Then the bicanonical map φ_2 of X is*

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1) *a morphism which embedds X as a projectively normal variety if X is of Type 1, 2, 3, 4, 5.2 or 6.2 (see Theorem 1.3 for the description of each type);*
2) *a birational morphism but not an embedding if X is of Type 9, 10, 11 or 12;*
3) *a finite morphism of degree 2 if X is of Type 5.1, 6.1, 7 or 8.2;*
4) *a finite morphism of degree 4 if X is of Type 8.1.*

The diverse behavior of the bicanonical maps exhibited in the above theorem is not seen in canonical covers of lower degrees and conjecturally does not happen for covers of all other degrees.

In Section 2, we deal with those types of quadruple Galois canonical covers for which $2K_X$ is very ample and embeds X as a projectively normal surface. Other than covers of the projective plane, the manifestation of such behavior can be seen in surfaces X with unbounded p_g but with bounded q. However, in the case of families with unbounded q, $2K_X$ is not very ample, even though the image is a projectively normal variety. One of the frequently used techniques to show that a 2–Veronese subring of a graded ring is generated in degree less than or equal to two is to first show that the ambient graded ring is generated in degree less than or equal to two. Formally one can construct graded rings in commutative algebra where this does not happen and yet the 2–Veronese subring is generated in degree 1. In this context, the quadruple covers of types 1, 2, 3, 4, 5.2 and 6.2 provide natural examples where the canonical ring is not generated in degree less than or equal to two, yet the 2–Veronese subrings are generated in degree less or equal one, thereby showing the normal generation of the bicanonical map.

Section 3 deals with those quadruple covers for which bicanonical maps are finite. The types for which the bicanonical maps are not birational have unbounded p_g and q. But the results of Xiao (see [Xia90, Theorems 1, 2 and 3]) say that if the bicanonical maps are not birational then they form bounded families with respect to p_g unless the surfaces possess a genus two pencil. In Section 3, we explicitly exhibit the genus two pencils that are indeed fibrations. So in addition to being a genus 3 fibration over $\mathbf{P}^1$, the families of types 5.1 and 6.1 are also genus two fibrations over an elliptic curve, and the families of type 7 are also genus 2 fibrations over a curve of genus m, where m takes on unbounded values.

Section 4 deals with the Types 9, 10, 11 and 12 where the image of the canonical map W is a singular surface of minimal degree. It was shown in [GP07] that in such a case $p_g \leq 4$ and $q = 0$. The behavior of the bicanonical map is very interesting for these types of surfaces: it is always birational but never an embedding. And, more interestingly, it is birational is two different ways: for the types 9, 10 or 12, $|2K_X|$ does not separate directions at the unique point $x \in \varphi^{-1}\{w\}$ (and is an isomorphism outside x). For type 11, $|2K_X|$ does not separate the two points x_1 and x_2 of $\varphi^{-1}\{w\}$, although φ_2 is locally an embedding at both of them (and it is an isomorphism outside x_1 and x_2). We prove these results by proving a non–vanishing theorem for certain ideal sheaves. To accomplish this, we construct explicit factorizations of some birational maps which are in half the cases crepant and in the other half not and handle the question of the non–vanishing result on a suitably constructed desingularization of X. In all of this, the algebra structure of the map p in (1.3.1) (the so–called desingularized diagram), which was precisely described in [GP07], also plays an important role.

The canonical ring of a surface of general type and the degrees of its generators have attracted interest among geometers for various reasons. One such reason is its applications to the study of Calabi–Yau threefolds. For example, results on ring generation are used in determining the very ampleness of line bundles on Calabi–Yau threefolds in the article [GP98] and have provided motivation in the construction of new examples of Calabi–Yau threefolds as can be seen in the work of Casnati [Cas06]. In Section 5 we prove a general result, Theorem 5.1, that gives a nice formula for the number of generators in degree 2 of the canonical ring of canonical covers, of arbitrary degree, of surfaces of minimal degree. The formula shows that the number of generators in degree 2 only

depends on the geometric and arithmetic genus of X. Theorem 5.2, which determines the generators of the canonical ring of quadruple covers, shows that there is no such formula for generators of degree 3 of the canonical ring of X, if X is an irregular surface of general type. In fact Theorem 5.2 shows that this number depends on the algebra structure of φ.

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1. Preliminaries and notation

We introduce the notation and the basic definitions that we will use throughout the article:

Convention. We work over an algebraically closed field of characteristic 0.

Notation 1.1. Throughout this article, unless otherwise stated, we will make the following assumptions and use the following notation:

(1) W will be an embedded projective algebraic surface of minimal degree, i.e., a surface such that $\deg W = \operatorname{codim} W + 1$.

(2) X will be a projective algebraic normal surface with at worst canonical singularities (that is, X is smooth or has rational double points; see [Bad01] for details about rational double points), whose canonical divisor K_X is ample and base–point–free.

(3) We will denote the canonical map of X as φ. Note that, by (2), φ is in fact a finite morphism.

(4) We will assume φ to be a Galois morphism of degree 4 whose image is a surface W of minimal degree, that is, $\varphi : X \longrightarrow W$ will be a *quadruple Galois canonical cover of a surface W of minimal degree*.

(5) We will denote the bicanonical map of X as φ_2.

We also recall the following standard notation, that will also be used throughout the article:

(6) By $\mathbf{F}_e$ we denote the Hirzebruch surface whose minimal section have self-intersection $-e$. If $e > 0$ let C_0 denote the minimal section of $\mathbf{F}_e$ and let f be one of the fibers of $\mathbf{F}_e$. If $e = 0$, C_0 will be a fiber of one of the families of lines and f will be a fiber of the other family of lines of $\mathbf{F}_0$.

(7) If a, b are integers such that $0 < a \leq b$, consider two disjoint linear subspaces $\mathbf{P}^a$ and $\mathbf{P}^b$ of $\mathbf{P}^{a+b+1}$. We denote by $S(a, b)$ the smooth rational normal scroll obtained by joining corresponding points of a rational normal curve in $\mathbf{P}^a$ and a rational normal curve of $\mathbf{P}^b$. Recall that $S(a, b)$ is the image of $\mathbf{F}_e$ by the embedding induced by the complete linear series $|C_0 + mf|$, with $a = m - e$, $b = m$ and $m > e + 1$.

If $a = b$, the linear series $|mC_0 + f|$ also gives a minimal degree embedding of $\mathbf{F}_0$, equivalent to the previous one by the automorphism of $\mathbf{P}^1 \times \mathbf{P}^1 = \mathbf{F}_0$ swapping the factors. In this case *our convention will always be to choose C_0 and f so that, when $W = S(m, m)$, W is embedded by $|C_0 + mf|$.*

If in addition $m = 1$, C_0 and f are indistinguishable in both $\mathbf{F}_0$ and $S(1, 1)$, so, *in such a case, for us C_0 will denote the fiber of any of the families of lines of $\mathbf{F}_0$ and f will denote the fiber of the other family.*

(8) If b is an integer, $b > 1$, consider a linear subspace $\mathbf{P}^b$ of $\mathbf{P}^{b+1}$. We denote by $S(0, b)$ the cone in $\mathbf{P}^{b+1}$ over a rational normal curve of $\mathbf{P}^b$. Recall that $S(0, b)$ is the image of $\mathbf{F}_e$ by the morphism induced by the complete linear series $|C_0 + mf|$, with $b = m = e$ and hence $e > 1$.

Remark 1.2. If $X \xrightarrow{\varphi} W$ is a Galois cover and W is smooth, then φ is flat.

We recall now the main features of the classification of φ, which was obtained in [GP08, Theorem 0.1] and in the main theorem of [GP07]. According to this classification φ falls into several different types described in the tables of the theorem below. We will refer to these types throughout the article.

Theorem 1.3. *Let $\varphi : X \longrightarrow W$ be as in Notation 1.1 (i.e., let φ be a quadruple Galois canonical cover of a surface W of minimal degree).*

1) *If W is* smooth, *then W is either linear $\mathbf{P}^2$ or a smooth Hirzebruch surface $\mathbf{F}_e$, with $0 \leq e \leq 2$, embedded by $|H|$, where $H = C_0 + mf$ $(m \geq e+1)$. Recall that the Galois group G of φ is either $\mathbf{Z}_4$ or $\mathbf{Z}_2^{\oplus 2}$.*

If $G = \mathbf{Z}_4$, then φ is the composition of two double covers $X_1 \xrightarrow{p_1} W$ branched along a divisor D_2 and $X \xrightarrow{p_2} X_1$, branched along the ramification of p_1 and $p_1^* D_1$, where D_1 is a divisor on W.

If $G = \mathbf{Z}_2^{\oplus 2}$, then X is the fiber product over W of two double covers of W branched along divisors D_1 and D_2 and φ is the natural morphism from the fiber product to W.

Moreover φ fits into one of the following types, determined by these characteristics:

Type	W	$p_g(X)$	G	$D_1 \sim$	$D_2 \sim$	$q(X)$
1	$\mathbf{P}^2$	3	$\mathbf{Z}_4$	*conic*	*quartic*	0
2	$\mathbf{P}^2$	3	$\mathbf{Z}_2^{\oplus 2}$	*quartic*	*quartic*	0
3	$S(m, m-e)$	$2m-e+2$	$\mathbf{Z}_4$	$(2m-e+1)f$	$4C_0 + (2e+2)f$	0
4	$S(m, m-e)$	$2m-e+2$	$\mathbf{Z}_2^{\oplus 2}$	$2C_0 + (2m+2)f$	$4C_0 + (2e+2)f$	0
5.1	$S(1,1)$	4	$\mathbf{Z}_4$	$6f$	$4C_0$	1
5.2	$S(m,m)$, $(m>1)$	$2m+2$	$\mathbf{Z}_4$	$(2m+4)f$	$4C_0$	1
6.1	$S(1,1)$	4	$\mathbf{Z}_2^{\oplus 2}$	$4C_0$	$2C_0 + 6f$	1
6.2	$S(m,m)$, $(m>1)$	$2m+2$	$\mathbf{Z}_2^{\oplus 2}$	$2C_0 + (2m+4)f$	$4C_0$	1
7	$S(m,m)$	$2m+2$	$\mathbf{Z}_2^{\oplus 2}$	$(2m+2)f$	$6C_0 + 2f$	m
8.1	$S(1,1)$	4	$\mathbf{Z}_2^{\oplus 2}$	$6f$	$6C_0$	4
8.2	$S(m,m)$, $(m>1)$	$2m+2$	$\mathbf{Z}_2^{\oplus 2}$	$(2m+4)f$	$6C_0$	$m+3$

2) *If W is* not smooth, *then $W = S(0,2)$ (a quadric cone in $\mathbf{P}^3$), X is regular and $\varphi : X \longrightarrow W$ fits into a commutative diagram (see* [GP07]*):*

(1.3.1)
$$\begin{tikzcd} \overline{X} \arrow[r, "\overline{\psi}"] \arrow[d, "p"] & X \arrow[d, "\varphi"] \\ Y \arrow[r, "\psi"] & W \end{tikzcd},$$

where ψ is the minimal desingularization of W, $\overline{X}$ is the normalization of the reduced part of $X \times_W Y$ and has at worst canonical singularities, and p and $\overline{\psi}$ are induced by the projections from the fiber product onto each factor, $\overline{\psi}$ being the morphism from $\overline{X}$ to its canonical model X. The structure of p is similar to the structure of φ described above for W smooth (see the main theorem of [GP07] *for details) and φ can be classified in another four types, determined by the following properties:*

Type	W	$p_g(X)$	G	$\overline{\psi}$ *is*	$q(X)$
9	$S(0,2)$	4	$\mathbf{Z}_2^{\oplus 2}$	*crepant*	0
10	$S(0,2)$	4	$\mathbf{Z}_4$	*crepant*	0
11	$S(0,2)$	4	$\mathbf{Z}_2^{\oplus 2}$	*non crepant*	0
12	$S(0,2)$	4	$\mathbf{Z}_4$	*non crepant*	0

2. Normal generation of the bicanonical bundle

In this section we study the types of quadruple Galois canonical covers X for which $2K_X$ is normally generated, i.e., we find out for what X the bicanonical morphism φ_2 embeds X as a projectively normal variety. As we will see in Section 4, φ_2 is never an embedding if W is singular, so throughout this section we assume W *to be smooth.*

Recall that by Remark 1.2, since we are assuming W to be smooth, φ is flat. Furthermore, the push down of $\mathscr{O}_X$ to W splits as an $\mathscr{O}_W$–module as follows:

$$\varphi_*\mathscr{O}_X = \mathscr{O}_W \oplus L_1^* \oplus L_2^* \oplus L_3^*, \tag{2.0.2}$$

with L_1, L_2 and L_3 and D_1 and D_2 of Theorem 1.3 satisfying the following properties (see [GP08, Remark 3.1], where the multiplicative structure that turns the second term of (2.0.2) into an $\mathscr{O}_W$–algebra is also described):

$$\begin{array}{lll} & L_1 \otimes L_2 = L_3 & \\ \text{if } G = \mathbf{Z}_2 \times \mathbf{Z}_2, & \text{then } L_1^* = \mathscr{O}_W(-1/2D_2) \text{ and} & L_2^* = \mathscr{O}_W(-1/2D_1); \text{ and} \\ \text{if } G = \mathbf{Z}_4, \text{ then} & L_1^* = \mathscr{O}_W(-1/2D_1 - 1/4D_2) & \text{and } L_2^* = \mathscr{O}_W(-1/2D_2). \end{array} \tag{2.0.3}$$

We will keep the notation introduced in (2.0.2) and (2.0.3) for the remaining of the article.

To prove or disprove the normal generation of $2K_X$ we will look at multiplication maps of global sections of line bundles on X. To study these maps we will use the $\mathscr{O}_W$–algebra structure of $\varphi_*\mathscr{O}_X$ as the following lemma explains:

Lemma 2.1. *Let A_1, A_2 be two line bundles on W and let $M_1 = \varphi^*A_1$ and $M_2 = \varphi^*A_2$ be their pull backs on X.*

Let

$$H^0(M_1) \otimes H^0(M_2) \xrightarrow{\beta} H^0(M_1 \otimes M_2)$$

be the multiplication map of global sections of M and N and let

$$\begin{array}{ccc} H^0(A_1) \otimes H^0(A_2) & \xrightarrow{\beta_1} & H^0(A_1 \otimes A_2), \\ H^0(A_1) \otimes H^0(A_2 \otimes L_1^*) & \xrightarrow{\beta_2} & H^0(A_1 \otimes A_2 \otimes L_1^*), \\ H^0(A_1) \otimes H^0(A_2 \otimes L_2^*) & \xrightarrow{\beta_3} & H^0(A_1 \otimes A_2 \otimes L_2^*), \\ H^0(A_1 \otimes L_1^*) \otimes H^0(A_2 \otimes L_2^*) & \xrightarrow{\beta_4} & H^0(A_1 \otimes A_2 \otimes L_3^*) \end{array}$$

be multiplication maps of global sections of line bundles on W.

1) *If β_1, β_2, β_3 and β_4 are surjective, so is β.*
2) *If $H^0(A_1 \otimes A_2 \otimes L_3^*) \neq 0$ but $H^0(A_i \otimes L_j^*) = 0$ for either*
   $i = 1$ and $j = 1, 2$, or
   $i = 2$ and $j = 1, 2$, or
   $i = 1, 2$ and $j = 1$, or
   $i = 1, 2$ and $j = 2$,

   then β is not surjective.

Proof. We have by projection formula

(2.1.1)
$$H^0(\varphi_* M_i) = H^0(A_i) \oplus H^0(A_i \otimes L_1^*) \oplus H^0(A_1 \otimes L_2^*) \oplus H^0(A_1 \otimes L_3^*) \text{ and}$$
$$H^0(\varphi_*(M_1 \otimes M_2)) = H^0(A_1 \otimes A_2) \oplus H^0(A_1 \otimes A_2 \otimes L_1^*) \oplus H^0(A_1 \otimes A_2 \otimes L_2^*) \oplus H^0(A_1 \otimes A_2 \otimes L_3^*).$$

The surjectivity of β is equivalent to the surjectivity of

$$H^0(\varphi_* M) \otimes H^0(\varphi_* N) \xrightarrow{\beta'} H^0(\varphi_*(M \otimes N)).$$

The $\mathscr{O}_W$–algebra structure of $\varphi_*\mathscr{O}_X$ is given by a multiplication map

$$\varphi_*\mathscr{O}_X \otimes \varphi_*\mathscr{O}_X \longrightarrow \varphi_*\mathscr{O}_X$$

which splits in several summands according to (2.0.2), as explained in [GP08, Remark 3.1]. From them, we are interested in the following four:

$$\begin{array}{lcl} \mathscr{O}_W \otimes \mathscr{O}_W & \xrightarrow{\simeq} & \mathscr{O}_W \\ \mathscr{O}_W \otimes L_1^* & \xrightarrow{\simeq} & L_1^* \\ \mathscr{O}_W \otimes L_2^* & \xrightarrow{\simeq} & L_2^* \\ L_1^* \otimes L_2^* & \xrightarrow{\simeq} & L_3^* \end{array} \tag{2.1.2}$$

The map β' also splits according to (2.1.1), so (2.1.2) implies that if $\beta_1, \beta_2, \beta_3$ and β_4 surject, all the summands of $H^0(\varphi_*(M_1 \otimes M_2))$ described in (2.1.1) are in the image of β', so β' and hence β are surjective. On the other hand, [GP08, Remark 3.1] also tells that the last map of (2.1.2) is the only summand mapping to L_3. Thus, under the hypotheses of 2), $H^0(A_1 \otimes A_2 \otimes L_3^*) \neq 0$ but it is not in the image of β', so β' and hence β are not surjective in this case. □

To apply Lemma 2.1 in proving the normal generation of $2K_X$ we will need this easy but useful observation that helps to handle multiplication maps of global sections:

Observation 2.2. *Let E and $L_1, ..., L_n$ be coherent sheaves on a variety X. Consider the map $H^0(E) \otimes H^0(L_1 \otimes \cdots \otimes L_r) \xrightarrow{\psi} H^0(E \otimes L_1 \otimes \cdots \otimes L_r)$ and the maps*

$$H^0(E) \otimes H^0(L_1) \xrightarrow{\alpha_1} H^0(E \otimes L_1)$$
$$H^0(E \otimes L_1) \otimes H^0(L_2) \xrightarrow{\alpha_2} H^0(E \otimes L_1 \otimes L_2)$$
$$\cdots\cdots$$
$$H^0(E \otimes L_1 \otimes \cdots \otimes L_{r-1}) \otimes H^0(L_r) \xrightarrow{\alpha_r} H^0(E \otimes L_1 \otimes \cdots \otimes L_r)$$

If $\alpha_1, ...\alpha_r$ are surjective then ψ is surjective.

Now we are ready to prove the normal generation of $2K_X$ for surfaces X of Types 1, 2, 3, 4, 5.2 and 6.2. As a warm–up for the rest, we start with the simplest case, that is, when $W = \mathbf{P}^2$:

Theorem 2.3. *Let $\varphi : X \longrightarrow W$ be a quadruple Galois canonical cover of Type 1 or 2. Then $2K_X$ is very ample and $|2K_X|$ embeds X as a projectively normal variety.*

Proof. Recall that $K_X = \varphi^*\mathscr{O}_{\mathbf{P}^2}(1)$. Since K_X is ample, the normal generation of $2K_X$ is equivalent to the surjectivity of

$$H^0(2K_X) \otimes H^0(2nK_X) \longrightarrow H^0((2n+2)K_X), \tag{2.3.1}$$

for all $n \geq 1$. We prove it in two steps.

Step 1. The first step is to show that

$$H^0(2K_X) \otimes H^0(2K_X) \xrightarrow{\beta} H^0(4K_X)$$

surjects. We know (see the structure of φ as described in Theorem 1.3) that, for both Types 1 and 2,

$$\varphi_* \mathscr{O}_X = \mathscr{O}_{\mathbf{P}^2} \oplus \mathscr{O}_{\mathbf{P}^2}(-2) \oplus \mathscr{O}_{\mathbf{P}^2}(-2) \oplus \mathscr{O}_{\mathbf{P}^2}(-4),$$

hence we have

$$\begin{aligned} H^0(\varphi_* 2K_X) &= H^0(\mathscr{O}_{\mathbf{P}^2}(2)) \oplus H^0(\mathscr{O}_{\mathbf{P}^2}) \oplus H^0(\mathscr{O}_{\mathbf{P}^2}) \quad \text{and} \\ H^0(\varphi_* 4K_X) &= H^0(\mathscr{O}_{\mathbf{P}^2}(4)) \oplus H^0(\mathscr{O}_{\mathbf{P}^2}(2)) \oplus H^0(\mathscr{O}_{\mathbf{P}^2}(2)) \oplus H^0(\mathscr{O}_{\mathbf{P}^2}). \end{aligned}$$

Then Lemma 2.1, 1) tells that in order to prove the surjectivity of β it is enough to show the surjectivity of the following multiplication maps on W:

$$\begin{aligned} H^0(\mathscr{O}_{\mathbf{P}^2}(2)) \otimes H^0(\mathscr{O}_{\mathbf{P}^2}(2)) &\xrightarrow{\beta_1} H^0(\mathscr{O}_{\mathbf{P}^2}(4)) \\ H^0(\mathscr{O}_{\mathbf{P}^2}) \otimes H^0(\mathscr{O}_{\mathbf{P}^2}(2)) &\xrightarrow{\beta_2} H^0(\mathscr{O}_{\mathbf{P}^2}(2)) \\ H^0(\mathscr{O}_{\mathbf{P}^2}) \otimes H^0(\mathscr{O}_{\mathbf{P}^2}) &\xrightarrow{\beta_4} H^0(\mathscr{O}_{\mathbf{P}^2}). \end{aligned}$$

The surjectivity of β_1 follows from the projective normality of the Veronese surface and the surjectivity of β_2 and β_4 is trivial.

Step 2. To complete the proof, we need to show that the multiplication map (2.3.1) surjects for all $n \geq 2$. In view of Observation 2.2, it is enough to show that

$$H^0(n'K_X) \otimes H^0(K_X) \longrightarrow H^0((n'+1)(K_X)),$$

for all $n' \geq 4$. Since K_X is ample and base–point–free, this follows from [Mum70, p. 41, Theorem 2] and the Kawamata–Viehweg vanishing theorem. $\square$

Now we go on to study the normal generation of $2K_X$ for those remaining cases for which W is smooth. In these cases W is a Hirzebruch surface, so in order to apply Lemma 2.1 we need to know first the surjectivity of certain multiplication maps of global sections of line bundles on Hirzebruch surfaces. This is done in the next lemma, where we give sufficient conditions for the surjectivity of such maps.

Lemma 2.4. *Let $W = \mathbf{F}_e, e = 0, 1$ or 2. Let $L_1 = a_1C_0 + b_1f$ and $L_2 = a_2C_0 + b_2f$ be two line bundles on W, with $a_i \geq 0$, $b_i \geq a_ie$. Then the multiplication map*

$$H^0(L_1) \otimes H^0(L_2) \xrightarrow{\mu} H^0(L_1 + L_2)$$

is surjective if in addition L_1 and L_2 satisfy one of the following conditions:

(a) $a_1 \geq 1, a_2 = na_1, b_2 = nb_1$ *with* $n \geq 1$, *and* $b_1 > a_1e$;
(b) $a_1 \geq 1$ *and* $a_2 \geq 2a_1 - 2 + e$ *if* $e \geq 1$ *or* $a_2 \geq 2a_1 - 1$ *if* $e = 0$, *and* $b_2 - b_1 \geq (a_2 - a_1)e - 1$;
(c) $a_1 > 0, a_2 = 0$;
(d) $a_1 = a_2 = 1, b_2 \geq b_1 - 1$;
(e) $W = \mathbf{F}_0$.

Proof. Observe that L_1 and L_2 are base–point–free because $a_i \geq 0$, $b_i \geq a_ie$.

We start proving the lemma under the assumption that (a) is satisfied. Observe that L_1 is ample because $a_1 \geq 1$ and $b_1 > a_1e$ by hypothesis. Then (a) is equivalent to the normal generation of L_1. We will apply [GP01, Theorem 1.3] for property N_0. Then we just need to show that $-K_X \cdot L_1 = (2C_0 + (2+e)f) \cdot (a_1C_0 + b_1f) \geq 3$. This amounts to showing $-a_1e + 2a_1 + 2b_1 \geq 3$, which is true by hypothesis. This proves (a).

To show the lemma when (b) is satisfied we use [Mum70, p. 41, Theorem 2]. Then it would be enough to show that $H^1(L_2 - L_1) = H^1((a_2 - a_1)C_0 + (b_2 - b_1)f)$ and $H^2(L_2 - 2L_1) = H^2((a_2 - 2a_1)C_0 + (b_2 - 2b_1)f)$ both vanish. Let $\pi : W \longrightarrow \mathbf{P}^1$ be the projection from W to $\mathbf{P}^1$. Note that by hypothesis $a_2 - a_1 \geq 0$, and $a_2 - 2a_1 \geq -1$, which implies $R^1\pi_*(L_2 - L_1) = 0$ and

$R^1\pi_*(L_2-2L_1)=0$. So $H^j(L_2-jL_1)=H^j(\pi_*(L_2-jL_1))$ for $j=1,2$. Since $a_2-a_1\geq 0$, then $\pi_*(L_2-L_1)$ splits as the sum of a_2-a_1+1 line bundles as follows

$$\pi_*(L_2-L_1)=\mathscr{O}_{\mathbf{P}^1}(b_2-b_1)\oplus\cdots\oplus\mathscr{O}_{\mathbf{P}^1}(b_2-b_1-(a_2-a_1)e). \tag{2.4.1}$$

Since by hypothesis $b_2-b_1\geq(a_2-a_1)e-1$, $H^1\pi_*(L_2-L_1)$, and hence $H^1(L_2-L_1)$, vanish. Since a curve has no second cohomology, $H^2(L_2-2L_1)$ also vanishes. This proves the lemma if (b) holds.

Now assume (c) holds. We will use Observation 2.2 repeatedly and apply [Mum70, p. 41, Theorem 2]. Note that, even though [Mum70, p. 41, Theorem 2] is stated for ample and base–point–free line bundles in [Mum70], it is still true if the line bundle is just base–point–free. By Observation 2.2 it suffices to prove that the multiplication map

$$H^0(a_1C_0+b_1'f)\otimes H^0(f)\xrightarrow{\mu'}H^0(a_1C_0+b_1'f)$$

is surjective for all $b_1'\geq b_1$. Now, since f is base–point–free, [Mum70, p. 41, Theorem 2] tells that to prove that μ' is surjective, it is enough to show $H^1(a_1C_0+(b_1'-1)f)=0$ and $H^2(a_1C_0+(b_1'-2)f)=0$. Since $a_1>0$, $R^1\pi_*(a_1C_0+(b_1'-1)f)$ and $R^1\pi_*(a_1C_0+(b_1'-2)f)$ both vanish and hence it is enough to prove the vanishing of $H^1(\pi_*(a_1C_0+(b_1'-1)f))$ and $H^2(\pi_*(a_1C_0+(b_1'-2)f))$. The latter cohomology vanishes because a curve has no second cohomology, and the former vanishes arguing as in (2.4.1) because $b_1'-1-a_1e\geq b_1-1-a_1e\geq -1$. This settles (c).

If (d) holds, the lemma follows directly from [Mum70, p. 41, Theorem 2] arguing as for (b) or (c), having in account that $R^1\pi_*((b_2-b_1)f)$ and $R^1\pi_*(-C_0+(b_2-2b_1)f)$ vanish and that $b_2-b_1\geq -1$ by hypothesis.

We will now assume that (e) holds. Without loss of generality we may assume $a_1>0$ since in this case $W=\mathbf{F}_0$ and by an automorphism of $\mathbf{F}_0$ we can interchange C_0 and f and rename a_2 as a_1 if need be. In view of Observation 2.2 it is enough to show that the multiplication maps

$$\begin{array}{rcll} H^0(a_1'C_0+b_1'f)\otimes H^0(f) & \xrightarrow{\mu_1} & H^0(a_1'C_0+(b_1'+1)f) & \text{and}\\ H^0(a_1'C_0+b_1'f)\otimes H^0(C_0) & \xrightarrow{\mu_2} & H^0((a_1'+1)C_0+b_1'f) & \end{array}$$

are surjective for all $a_1'\geq a_1$ and all $b_1'\geq b_1$. The maps μ_1 are surjective because (c) holds in this case. The maps μ_2 are also surjective because (c) also holds if $b_1>0$, after applying an automorphism of $\mathbf{F}_0$ interchanging C_0 and f. Finally, if $b_1=0$, after the automorphism of $\mathbf{F}_0$, μ_2 becomes

$$H^0(a_1'f)\otimes H^0(f)\xrightarrow{\mu_2}H^0((a_1'+1)f).$$

Now by [Mum70, p. 41, Theorem 2] it is enough to show that $H^1((a_1'-1)f)=0$ and $H^2((a_1'-2)f)=0$. Since $R^1\pi_*sf$ vanishes for any $s\in\mathbf{Z}$ and $a_1'-1\geq a_1-1\geq 0$, arguing as for (b), (c) or (d) we conclude the surjectivity of μ_2. □

Now we turn our attention to the bicanonical morphism of quadruple Galois canonical covers X of smooth rational normal scrolls when X is regular.

Theorem 2.5. *Let $\varphi: X\longrightarrow W$ be a quadruple Galois canonical cover of Type 3 or 4. Then $2K_X$ is very ample and $|2K_X|$ embeds X as a projectively normal variety.*

Proof. Recall that $K_X=\varphi^*(C_0+mf)$ (see Theorem 1.3). As in the proof of Theorem 2.3, since K_X is ample, the normal generation of $2K_X$ is equivalent to the surjectivity of

$$H^0(2K_X)\otimes H^0(2nK_X)\longrightarrow H^0((2n+2)K_X), \tag{2.5.1}$$

for all $n\geq 1$. We prove so in two steps.

Step 1. The first step is to show that

$$H^0(2K_X)\otimes H^0(2K_X)\xrightarrow{\beta}H^0(4K_X)$$

surjects and we argue as in Step 1 of the proof of Theorem 2.3. We know (see the structure of φ as described in Theorem 1.3) that, for both Types 3 and 4,

$$\varphi_*\mathscr{O}_X = \mathscr{O}_W \oplus \mathscr{O}_W(-C_0-(m+1)f) \oplus \mathscr{O}_W(-2C_0-(e+1)f) \oplus \mathscr{O}_W(-3C_0-(m+e+2)f),$$

hence we have

$$\begin{aligned} H^0(\varphi_*2K_X) &= H^0(2C_0+2mf) \oplus H^0(C_0+(m-1)f) \oplus H^0((2m-e-1)f) \quad \text{and} \\ H^0(\varphi_*4K_X) &= H^0(4C_0+4mf) \oplus H^0(3C_0+(3m-1)f) \oplus H^0(2C_0+(4m-e-1)f) \oplus \\ & \quad H^0(C_0+(3m-e-2)f). \end{aligned}$$

Then Lemma 2.1, 1) tells that in order to prove the surjectivity of β it is enough to show the surjectivity of the following multiplication maps on W:

$$\begin{aligned} H^0(2C_0+2mf) \otimes H^0(2C_0+2mf) &\xrightarrow{\beta_1} H^0(4C_0+4mf) \\ H^0(2C_0+2mf) \otimes H^0(C_0+(m-1)f) &\xrightarrow{\beta_2} H^0(3C_0+(3m-1)f) \\ H^0(2C_0+2mf) \otimes H^0((2m-e-1)f) &\xrightarrow{\beta_3} H^0(2C_0+(4m-e-1)f) \\ H^0(C_0+(m-1)f) \otimes H^0((2m-e-1)f) &\xrightarrow{\beta_4} H^0(C_0+(3m-e-2)f). \end{aligned}$$

To prove the surjectivity of $\beta_1, \beta_2, \beta_3$ and β_4 we use Lemma 2.4. Recall that $m \geq e+1$ and $0 \leq e \leq 2$. Then multiplication map β_1 is surjective by Lemma 2.4, (a), β_2 by Lemma 2.4, (b), and β_3 and β_4 by Lemma 2.4, (c).

Step 2. To complete the proof, we need to show that the multiplication map (2.5.1) surjects for all $n \geq 2$. This follows from the same argument used for Step 2 of the proof of Theorem 2.3. □

Now we focus on the cases in which X is irregular. Under this hypothesis, in the next theorem we find out when $2K_X$ is normally generated:

Theorem 2.6. *Let $\varphi : X \longrightarrow W$ be a quadruple Galois canonical cover of Type 5.2 or 6.2. Then $2K_X$ is very ample and $|2K_X|$ embeds X as a projectively normal variety.*

Proof. As we observed in the proof of Theorems 2.3 and 2.5, the normal generation of $2K_X$ is equivalent to the surjectivity of

$$H^0(2K_X) \otimes H^0(2nK_X) \longrightarrow H^0((2n+2)K_X), \tag{2.6.1}$$

for all $n \geq 1$. We prove the surjectivity of (2.6.1) in two steps:

Step 1: We show that

$$H^0(2K_X) \otimes H^0(2K_X) \xrightarrow{\beta} H^0(4K_X)$$

surjects. To see this we argue as in Step 1 of the proofs of Theorems 2.3 and 2.5. Recall that $K_X = \varphi^*(C_0+mf)$ (see Theorem 1.3). We know (see the structure of φ as described in Theorem 1.3)

$$\varphi_*\mathscr{O}_X = \mathscr{O}_W \oplus \mathscr{O}_W(-C_0-(m+2)f) \oplus \mathscr{O}_W(-2C_0) \oplus \mathscr{O}_W(-3C_0-(m+2)f),$$

hence we have

$$\begin{aligned} H^0(\varphi_*2K_X) &= H^0(2C_0+2mf) \oplus H^0(C_0+(m-2)f) \oplus H^0((2mf) \quad \text{and} \\ H^0(\varphi_*4K_X) &= H^0(4C_0+4mf) \oplus H^0(3C_0+(3m-2)f) \oplus H^0(2C_0+4mf) \oplus \\ & \quad H^0(C_0+(3m-2)f). \end{aligned}$$

Then Lemma 2.1, 1) tells that, in order to prove the surjectivity of β, it is enough to show the surjectivity of the following multiplication maps on W:

$$\begin{array}{rcl} H^0(2C_0+2mf)\otimes H^0(2C_0+2mf) & \xrightarrow{\beta_1} & H^0(4C_0+4mf) \\ H^0(2C_0+2mf)\otimes H^0(C_0+(m-2)f) & \xrightarrow{\beta_2} & H^0(3C_0+(3m-2)f) \\ H^0(2C_0+2mf)\otimes H^0(2mf) & \xrightarrow{\beta_3} & H^0(2C_0+4mf) \\ H^0(C_0+(m-2)f)\otimes H^0(2mf) & \xrightarrow{\beta_4} & H^0(C_0+(3m-2)f) \end{array}$$

To prove the surjectivity of $\beta_1, \beta_2, \beta_3$ and β_4 we use Lemma 2.4. Recall that $m \geq 2$ and $e = 0$. Then $\beta_1, \beta_2, \beta_3$ and β_4 surject by Lemma 2.4, (e).

Step 2: To complete the proof, we need to show that the multiplication map (2.6.1) surjects for all $n \geq 2$. This follows form the same argument used for Step 2 of the proof of Theorem 2.3. $\square$

To end the study of whether $2K_X$ is normally generated or not when W is smooth, we look at the remaining types in which X is irregular, i.e, Types 5.1, 6.1, 7, 8. As the following remark shows, for none of these types is $2K_X$ normally generated:

Remark 2.7. *Let $\varphi : X \longrightarrow W$ be a quadruple Galois canonical cover of Type 5.1, 6.1, 7 or 8. Then $|2K_X|$ does not embed X as a projectively normal variety.*

Proof. If $|2K_X|$ embedded X as a projectively normal variety, in particular it would embed X as a quadratically normal variety, so the multiplication map

$$H^0(2K_X)\otimes H^0(2K_X) \xrightarrow{\beta} H^0(4K_X)$$

should surject. Recall that $K_X = \varphi^*(C_0+mf)$ (see Theorem 1.3). We are going to apply Lemma 2.1, 2). If X is of type 5.1 or 6.1, then the line bundle L_2^* of (2.0.2) is $\mathscr{O}_W(-C_0-3f)$ (see Theorem 1.3), so $2C_0+2mf-L_2 = C_0-f$ (recall that for Types 5.1 and 6.1, $m=1$). If X is of type 7, the line bundle L_2^* of (2.0.2) is $\mathscr{O}_W(-3C_0-f)$ (see Theorem 1.3), so $2C_0+2mf-L_2 = -C_0+(2m-1)f$. Finally, if X is of type 8, the line bundle L_2^* of (2.0.2) is $\mathscr{O}_W(-3C_0)$ (see Theorem 1.3), so $2C_0+2mf-L_2 = -C_0+2mf$. In any case, $H^0(2C_0+2mf-L_2)=0$ for Types 5.1, 6.1, 7 and 8. On the other hand, $4C_0+4mf-L_2$ is $3C_0+f$, $C_0+(4m-1)f$ and C_0+4mf for Types 5.1 and 6.1, Type 7 and Type 8 respectively, so in all cases $H^0(4C_0+4mf-L_2)\neq 0$. Then Lemma 2.1, 2) implies that β does not surject. $\square$

3. The bicanonical morphism when X is irregular

We have seen in the previous section that if W is smooth and X is regular, then φ_2 embeds X as a projectively normal variety; in particular, $2K_X$ is very ample. We also saw that if X is irregular, then, in some cases φ_2 embeds X as a projectively normal variety (Types 5.2 and 6.2) and in others (Types 5.1, 6.1, 7 and 8) does not (see Theorem 2.6 and Remark 2.7). For the latter cases, we will prove in this section that φ_2 is not even an embedding, we will compute its degree and we will describe $\varphi_2(X)$. Thus in this section X will be irregular and, by Theorem 1.3, $W = \mathbf{F}_0$.

Theorem 3.1. *Let $\varphi : X \longrightarrow W$ be a quadruple Galois canonical cover of Type 5.1, 6.1 or 7. Recall that $K_X = \varphi^*H$ ($H = C_0+mf$; $m=1$ for Types 5.1 and 6.1). Then*

(1) *the morphism φ_2 is a $2:1$;*
(2) *the image of φ_2 is a ruled surface of irregularity m. More precisely, $\varphi_2(X) = G\times D$, where $G = \mathbf{P}^1$ and D is a smooth curve of genus m ($m=1$ for Types 5.1 and 6.1), embedded by $|2D+4mG|$ (G is embedded as a conic and D is embedded as a projectively normal curve of degree $4m$).*

Proof. First we deal with φ of Types 6.1 and 7. Recall that if φ is of Type 6.1, then it is the fiber product of two double covers of W branched along $D_1 \sim 4C_0$ and $D_2 \sim 2C_0 + 6f$ respectively and if φ is of Type 7, then it is the fiber product of two double covers of W branched along $D_1 \sim (2m+2)f$ and $D_2 \sim 6C_0 + 2f$ respectively. Then φ factors as $\varphi = p_1 \circ p_2$, where $p_1 : X' \longrightarrow W$ is the double cover of W branched along D_1 and $p_2 : X \longrightarrow X'$ is the double cover of X' branched along $p_1^* D_2$. Thus $\varphi_* \mathscr{O}_X$ splits as

$$\varphi_* \mathscr{O}_X = \mathscr{O}_W \oplus \mathscr{O}_W(-2C_0) \oplus \mathscr{O}_W(-C_0 - 3f) \oplus \mathscr{O}_W(-3C_0 - 3f) \tag{3.1.1}$$

if φ is of Type 6.1 and

$$p_{1*} \mathscr{O}_{X'} = \mathscr{O}_W \oplus \mathscr{O}_W(-2C_0). \tag{3.1.2}$$

is the subalgebra of $\varphi_* \mathscr{O}_X$ that corresponds to p_1 in this case. Likewise, if φ is of Type 7,

$$\varphi_* \mathscr{O}_X = \mathscr{O}_W \oplus \mathscr{O}_W(-(m+1)f) \oplus \mathscr{O}_W(-3C_0 - f) \oplus \mathscr{O}_W(-3C_0 - (m+2)f). \tag{3.1.3}$$

and the subalgebra of $\varphi_* \mathscr{O}_X$ that corresponds to p_1 is

$$p_{1*} \mathscr{O}_{X'} = \mathscr{O}_W \oplus \mathscr{O}_W(-(m+1)f). \tag{3.1.4}$$

Since $K_X = \varphi^* H$, it follows from the projection formula and from (3.1.1), (3.1.2), (3.1.3) and (3.1.4) that the global sections of $2K_X$ can be identified with the global sections of $p_{1*} \mathscr{O}_{X'} \otimes 2H$, so φ_2 factors through X'. More precisely, $\varphi_2 = \varphi_2' \circ p_2$, where φ_2' is induced by the complete linear series of $L = p_1^*(2H)$ ($m = 1$ for Type 6.1).

Now let us study X'. The structure of D_1 implies that X' is the product of $G = \mathbf{P}^1$ and a smooth curve D. For Type 6.1, D is the pullback of f and has genus 1. For Type 7, D is the pullback of C_0 and has genus m. Using the projection formula and the Leray spectral sequence, it follows easily that $H^1(L - p_1^* C_0) = H^1(L - p_1^* f) = 0$, hence $|L|$ restricts to a complete linear series both on D and on G. For Type 6.1, the restriction of L to G has degree 2 and the restriction of L to D has degree 4, so φ_2' embeds X' by $|2D + 4G|$ and its image is the product of a smooth conic and a smooth, projectively normal elliptic curve of degree 4. In particular φ_2 is $2:1$.

Now, for Type 7, the restriction of L to G has degree 2 as before, so $|L|$ maps G onto a smooth conic. On the other hand the restriction of L to D has degree $4m$, so $|L|$ embeds D as projectively normal curve of degree $4m$. Summarizing, φ_2 is $2:1$, φ_2' is the embedding of X' by the complete linear series $|2D + 4mG|$ and the image of φ_2' (which is the same as the image of φ_2) is the product of a smooth conic and a projectively normal curve of genus m and degree $4m$.

Now we deal with Type 5.1. Recall (see Theorem 1.3) that φ factors through a double cover $p_1 : X' \longrightarrow W$, branched along $D_2 \sim 4C_0$. From the structure of φ described in Theorem 1.3 it follows also that $\varphi_* \mathscr{O}_X$ splits as (3.1.1) and that the subalgebra of $\varphi_* \mathscr{O}_X$ corresponding to p_1 is like the one described in (3.1.2). Then, arguing as for Type 6.1 we see that φ_2 factors through p_1. Note now that the branch divisor of $p_1 : X' \longrightarrow W$ is linearly equivalent to $4C_0$ for both Type 5.1 and 6.1. Thus the rest of the argument for Type 5.1 is the same as for Type 6.1. □

Theorem 3.2. *Let $\varphi : X \longrightarrow W$ be a quadruple Galois canonical cover of Type 8. Then*

(1) *the surface X is the product of a smooth curve of genus 2 and a smooth curve of genus $m+1$;*

(2.1) *if $m = 1$ (i.e, $\varphi : X \longrightarrow W$ is of Type 8.1), then the morphism φ_2 is $4:1$ and its image is W;*

(2.2) *if $m > 1$ (i.e, $\varphi : X \longrightarrow W$ is of Type 8.2), then φ_2 is $2:1$;*

(3.1) *if $m = 1$, then the image of φ_2 is $\mathbf{F}_0$, embedded by $|2C_0 + 2f|$;*

(3.2) *if $m > 1$, then the image of φ_2 is the product of a smooth, projectively normal curve of genus $m+1$ and degree $4m$ and a smooth conic.*

Proof. Recall that φ is the fiber product of two double covers of W, branched along $D_1 \sim (2m+4)f$ and $D_2 \sim 6C_0$ respectively. This implies that $X = G \times D$, where G is a smooth curve of genus 2 (the reduced structure of the pullback of a fiber f by φ at the branch locus of φ) and D is a smooth curve of genus $m+1$ (the reduced structure of the pullback of C_0 by φ at the branch locus of φ). We will show that $|2K_X|$ restricts to a complete linear series both on G and on D. For this it suffices to show that $H^1(2K_X - G) = H^1(2K_X - D) = 0$. To see $H^1(2K_X - G) = 0$ we use Kodaira vanishing, noting that G is numerically equivalent to $\frac{1}{2}\varphi^* f$, so $K_X - G$ is numerically equivalent to $\varphi^*(C_0 + (m - \frac{1}{2}f))$, hence ample. To see $H^1(2K_X - D) = 0$ we argue similarly, noting that D is numerically equivalent to $\frac{1}{2}\varphi^* C_0$. Then $\varphi_2(X)$ is the product of $\varphi_2(G)$ and $\varphi_2(D)$.

For general f, $\varphi^{-1}f$ is the disjoint union of two curves, each algebraically equivalent to G. Thus $G^2 = 0$. Likewise, for general C_0, $\varphi^{-1}C_0$ is the disjoint union of two curves, each algebraically equivalent to D, so $D^2 = 0$. Then $2K_X|_G = 2K_G$ and $2K_X|_D = 2K_D$. Since the genus of G is 2, then $\varphi_2|_G$ is $2:1$ and its image is a smooth conic. Since the genus of D is $m+1$, if $m > 1$, then $\varphi_2|_D$ is an embedding, and if $m = 1$, then $\varphi_2|_D$ is $2:1$ and its image is a smooth conic. Thus if $m = 1$, φ_2 is $4:1$ and its image is $\mathbf{F}_0$, embedded by $|2C_0 + 2f|$ in $\mathbf{P}^8$. Finally, if $m > 1$, then φ_2 is $2:1$ and its image is ruled surface of irregularity $m+1$; precisely, the product of a smooth, projectively normal curve of genus $m+1$ and degree $4m$ and a smooth conic. This completes the proof. □

Remark 3.3. To prove Theorem 3.2 we could have also argued as in the proof of Theorem 3.1, as we outline now. Let us call as $p_1 : X' \longrightarrow W$ the double cover of W branched along D_1. Note that

$$\varphi_* \mathscr{O}_X = \mathscr{O}_W \oplus \mathscr{O}_W(-(m+2)f) \oplus \mathscr{O}_W(-3C_0) \oplus \mathscr{O}_W(-3C_0 - (m+2)f). \tag{3.3.1}$$

and that

$$p_{1*}\mathscr{O}_{X'} = \mathscr{O}_W \oplus \mathscr{O}_W(-(m+2)f) \tag{3.3.2}$$

is the subalgebra that corresponds to p_1. Then the projection formula implies that the global sections of $2K_X$ can be identified with the global sections of $2H \otimes (\mathscr{O}_W \oplus \mathscr{O}_W(-(m+2)f))$. Moreover, for Type 8.1, the global sections of $2K_X$ can be identified with the global sections of $2H$, since in this case $m = 1$. This implies that for Type 8.1, φ_2 factors through φ (and it is therefore $4:1$) and for Type 8.2, φ_2 factors through p_1. Now the argument would follow the same lines as the proof of Theorem 3.1. □

Theorems 3.1 and 3.2 say in particular that if X is of Type 5.1, 6.1, 7 or 8, φ_2 is not birational. There is a good reason for this to happen, namely, the fact that in these cases X has a pencil of curves of genus 2. In the next remarks we observe that, indeed, if X has a pencil of curves of genus 2, φ_2 cannot be birational and we explicitly describe the pencil of curves of genus 2 if X is of Type 5.1, 6.1, 7 or 8.

Remark 3.4. *If X has a pencil of genus 2 curves, then this pencil is base–point–free and φ_2 is not birational.*

Proof. The pencil of genus 2 curves is base–point–free because of our hypothesis of K_X being ample and base–point–free. Then, if C is a general member of the pencil, $|2K_X|$ restricts to a linear base–point–free subseries of $|K_C|$, hence to the complete canonical linear series of C. Then φ_2 maps C two–to–one onto its image. □

Remark 3.5. *We exhibit explicitly a pencil of genus 2 curves on X of Type 5.1, 6.1, 7 and 8. In particular*

1. *if X is of Type 5.1 or Type 6.1, X possesses an elliptic pencil of curves of genus 2;*
2. *if X is of Type 7, X possesses a genus m pencil of curves of genus 2;*
3. *if X is of Type 8, X possesses a genus $m+1$ pencil of curves of genus 2;*

Proof. If X is of Type 5.1, 6.1, 7 we saw in the proof of Theorem 3.1 that $\varphi = p_1 \circ p_2$, where $p_1 : X' \longrightarrow W$ and $p_2 : X \longrightarrow X'$ are double covers, $X' = D \times \mathbf{P}^1$ and D is a smooth curve of genus 1 if X is of Type 5.1 or 6.1, and a smooth curve of genus m if X is of Type 7. Then p_2 is branched along a divisor meeting a general fiber of X' at 6 six distinct points (see Theorem 1.3), so X is a fibration over D whose general fiber is a smooth curve of genus 2.

If X is of Type 8, we proved in Theorem 3.2 that X is the product of a smooth curve of genus 2 and a smooth curve of genus $m+1$, so obviously X possesses a genus $m+1$ base–point–free pencil of curves of genus 2.

Note that the fibration above comes from the Stein factorization of g, where g is the composition of φ followed by the projection of W onto f if X is of Type 5.1 or 6.1, and by the projection of W onto C_0 if X is of Type 7 or 8. □

Remark 3.6. The existence of pencils of genus 2 curves if X is of Type 5.1, 6.1, 7 or 8 follows also indirectly from more general results on the classification of the bicanonical maps of surfaces of general type, stated in [Xia90], having in account what the degree and how the image of φ_2 are (see Theorems 3.1 and 3.2).

Indeed; if X is of Type 5.1, 6.1, 7 or 8.2, then Theorems 3.1 and 3.2 say that φ_2 has degree 2. If X did not have a pencil of genus 2 curves, [Xia90, Theorem 2] would imply that either $q(X) = 0$ or $\varphi_2(X)$ is rational. However Theorem 1.3 says that $q(X) > 0$. On the other hand, Theorems 3.1 and 3.2 say that the image of φ_2 is a non–rational ruled surface, so X should have a pencil of genus 2 curves.

If X is of Type 8.1, Theorem 1.3 implies that X does not fit in the list of exceptions of [Xia90, Theorem 1]. On the other hand Theorem 3.2 says that φ_2 has degree 4, hence [Xia90, Theorem 1] implies that X should have a pencil of genus 2 curves.

In Theorems 3.1 and 3.2 we have described the image of φ_2 for X of each of Types 5.1, 6.1, 7 and 8. We settle now the question of whether these images are projectively normal varieties:

Corollary 3.7. *Let $\varphi : X \longrightarrow W$ be a quadruple Galois canonical cover of Type 5.1, 6.1, 7 or 8. The image of X by φ_2 is a projectively normal variety.*

Proof. If φ is of Type 8.1, $\varphi_2(X)$ is $\mathbf{F}_0$ embedded by $|2C_0+2f|$, which is well known to be projectively normal (it does even satisfy property N_5; see [GP01, Theorem 1.3]). For the remaining types recall that $\varphi_2(X) = \varphi_2'(X')$, where $p_1 : X' \longrightarrow W$ is a double cover branched along a divisor linearly equivalent to $4C_0$ in Types 5.1 and 6.1, to $(2m+2)f$, in Type 7 and to $(2m+4)f$ in Type 8.2. Recall also that φ_2' is induced by $|L|$, where $L = p_1^*(2H)$. Since L is ample, to prove that $|L|$ embeds X' as a projectively normal variety it suffices to show that the multiplication maps

$$H^0(L) \otimes H^0(rL) \longrightarrow H^0((r+1)L)$$

are surjective for all $r \geq 1$. Arguing as in Section 2 it follows from (3.1.2), (3.1.4) and (3.3.2) that it suffices to check that the following multiplication maps on W surject:

$$\begin{aligned} H^0(2C_0+2f) \otimes H^0(2rC_0+2rf) &\longrightarrow H^0((2r+2)C_0+(2r+2)f) \quad \text{and} \\ H^0(2f) \otimes H^0(2rC_0+2rf) &\longrightarrow H^0((2rC_0+(2r+2)f), \end{aligned} \tag{3.7.1}$$

for Types 5.1 and 6.1;

$$\begin{aligned} H^0(2C_0+2mf) \otimes H^0(2rC_0+2rmf) &\longrightarrow H^0((2r+2)C_0+(2r+2)mf) \quad \text{and} \\ H^0(2C_0+(m-1)f) \otimes H^0(2rC_0+2rf) &\longrightarrow H^0((2rC_0+(2r+m-1)f), \end{aligned} \tag{3.7.2}$$

for Type 7; and

$$\begin{aligned} & H^0(2C_0+2mf)\otimes H^0(2rC_0+2rmf) \longrightarrow H^0((2r+2)C_0+(2r+2)mf) \quad \text{and} \\ & H^0(2C_0+(m-2)f)\otimes H^0(2rC_0+2rf) \longrightarrow H^0((2rC_0+(2r+m-2)f), \end{aligned} \tag{3.7.3}$$

$(m>1)$, for Type 8. By Observation 2.2, to prove the surjectivity of the maps (3.7.1), (3.7.2) and (3.7.3) it suffices to see that the multiplication maps on W

$$\begin{aligned} & H^0(s_1C_0+s_2f)\otimes H^0(C_0) \longrightarrow H^0((s_1+1)C_0+s_2f) \quad \text{and} \\ & H^0(s_1C_0+s_2f)\otimes H^0(f) \longrightarrow H^0(s_1C_0+(s_2+1)f) \end{aligned} \tag{3.7.4}$$

surject for all $s_1, s_2 \geq 2$. This follows from [Mum70, p. 41, Theorem 2]. □

4. The bicanonical morphism when W is singular

In this section we prove that if W is singular (and hence, see [GP07], $W=S(0,2)$) then $2K_X$ is not very ample. We also see that, in constrast with the case when W is smooth (in which either φ_2 is an embedding with projectively normal image or φ_2 has degree bigger than 1) if W is singular, then φ_2 is birational but not an embedding. Precisely, we have this

Theorem 4.1. *Let $\varphi : X \longrightarrow W$ be a quadruple Galois canonical cover of $W=S(0,2)$ and let w the vertex of W.*

Then φ_2 is a birational morphism but $2K_X$ is not very ample because φ_2 fails to be an embedding at $\varphi^{-1}\{w\}$. Precisely,

1. *if φ is of Type 9, 10 or 12, then $|2K_X|$ does not separate directions at the unique point $x \in \varphi^{-1}\{w\}$;*
2. *if φ is of Type 11 then $|2K_X|$ does not separate the two points x_1 and x_2 of $\varphi^{-1}\{w\}$, although φ_2 is locally an embedding at both of them.*

Moreover, φ_2 is an embedding on the set of smooth points of $X-\varphi^{-1}\{w\}$ and, if X has the mildest possible singularities (see [GP07, Corollary 5.1 and Propositions 5.2, 5.3 and 5.4]*), then φ_2 is an embedding outside $\varphi^{-1}\{w\}$.*

Before we prove Theorem 4.1 we state a couple of lemmas.

Lemma 4.2. *Let S be an irreducible, normal surface with only rational singularities, let $\pi : \widetilde{S} \longrightarrow S$ be a proper birational map and let $\mathfrak{m}$ be a maximal ideal sheaf on S. Then $\pi_*(\mathfrak{m}^n\mathscr{O}_{\widetilde{S}})=\mathfrak{m}^n$.*

Proof. The lemma follows from Remark c) of Section 5, Theorem 7.1 and Proposition 6.2 of [Lip69]. □

Lemma 4.3. *Let S be a regular surface of general type whose canonical map is a degree n morphism $\psi : S \longrightarrow W$ onto a surface of minimal degree W. Let $C \in |K_S|$ be a smooth, irreducible curve. If C is hyperelliptic, then $n=2$.*

Proof. Since K_S is base–point–free, then $K_S|_C$ is a base–point–free theta–characteristic on C. Let g be the genus of C. Thus, if C is hyperelliptic, $K_S|_C$ is $\frac{g-1}{2}$ times the g^1_2 of C and $h^0(K_S|_C)=\frac{g+1}{2}$. On the other hand, since S is regular, $h^0(K_S|_C)=p_g-1$. Since W is a surface of minimal degree, $g-1=\deg K_S|_C=K_S^2=n(p_g-2)$, hence $g-1=n(p_g-2)=n(\frac{g+1}{2}-1)$, so $n=2$. □

Proof of Theorem 4.1. We use the notation of 1.3.1. We first prove that if X has the mildest possible singularities, then φ_2 is an embedding outside $\varphi^{-1}\{w\}$. Recall that outside w, Y and W are isomorphic. Recall also that outside $\varphi^{-1}\{w\}$, $\overline{X}$ and X are isomorphic and p and φ are equal. Let now x_1 and x_2 be two distinct points of $X-\varphi^{-1}\{w\}$. If $\varphi(x_1)\neq\varphi(x_2)$, then $\varphi_2(x_1)\neq\varphi_2(x_2)$, since $\varphi_2(X)$ can be projected to a 2–Veronese reembedding of W. Thus, let us assume $\varphi(x_1)=\varphi(x_2)$

and let us call y the inverse image of $\varphi(x_1) = \varphi(x_2)$ by q. Consider the linear system $|C_0 + 2f|$ in Y. Since it is base–point–free and big, there exists a smooth connected curve in $|C_0 + 2f|$, not meeting C_0, passing through y and meeting the branch locus of p in such a way that its pullback C by p is smooth and irreducible. Then $\overline{q}(C)$ is isomorphic to C and belongs to $|K_X|$, is smooth and irreducible and passes through x_1 and x_2. Since φ has degree 4, C is non hyperelliptic by Lemma 4.3. Since X is regular, $|2K_X|$ restricts to the complete canonical series of C, thus $|2K_X|$ embeds C and therefore separates x_1 and x_2. A similar argument proves that in general φ_2 separates smooth points of $\overline{X} - \varphi^{-1}\{w\}$. Moreover, this argument can be adapted to show that φ_2 separates directions in the set of smooth points of $\overline{X} - \varphi^{-1}\{w\}$.

Now we see what happens at $\varphi^{-1}\{w\}$. First we treat the case in which $\varphi^{-1}\{w\}$ consists of only one point x (Types 9, 10 and 12). Let $\mathscr{O}_{X,x}$ be the local ring at x, let $\mathfrak{m}_x$ be the maximal ideal of x and let

$$Z = \text{ Spec } \mathscr{O}_{X,x}/\mathfrak{m}_x^2$$

be the first infinitesimal neighbourhood of x. We want to prove that $|2K_X|$ does not separates directions at x. This is equivalent to proving that

$$H^0(2K_X) \longrightarrow H^0(2K_X|_Z)$$

is not surjective. Since, by Kodaira vanishing $H^1(2K_X) = 0$, the latter is equivalent to the non vanishing of $H^1(2K_X \otimes \mathfrak{m}_x^2)$. To study this cohomology group we will use Lemma 4.2. Thus we will compute $H^1(2K_X \otimes \mathfrak{m}_x^2)$ by arguing on $\overline{X}$ and with $L = \overline{q}^*K_X = p^*(C_0 + 2f)$. We denote $F = p^{-1}C_0$.

Now we argue for each Type 9, 10 and 12. We start with Type 12. In this case (see [GP07, Corollary 5.1]) the morphism $\overline{X} \xrightarrow{\overline{q}} X$ is the contraction of the smooth line F. The line F consists of smooth points of $\overline{X}$ and an A_1 singularity $\overline{x}$. Recall also that the self–intersection of F is $F^2 = -\frac{1}{2}$ and that $p^*C_0 = 4F$. Thus $\overline{q}$ factorizes as a composition of blowing ups and blowing downs in the following way: Let $g : \widehat{X} \longrightarrow \overline{X}$ be the blowing up of $\overline{X}$ at $\overline{x}$. Then the exceptional divisor F_1 of g has $F_1^2 = -2$ and the strict transform of F is a line F_2 with $F_2^2 = -1$. Then we obtain X from $\widehat{X}$ by contracting first F_2 and then F_1, so $\widehat{X}$ is obtained from X by performing two consecutive blowing ups, $g_1 : X' \longrightarrow X$ and $g_2 : \widehat{X} \longrightarrow X'$, the first one centered at x. Let us call $M = g_2^*(g_1^*K_X)$ and recall that $L = \overline{q}^*K_X = p^*(C_0 + 2f)$. Notice also that $M = g^*L$. Then, local computations and Lemma 4.2 yield

$$(g_2 \circ g_1)_*\mathscr{O}_{\widehat{X}}(-2F_1 - 2F_2) = \mathfrak{m}_x^2.$$

Then, by projection formula,

$$(g_2 \circ g_1)_*(2M \otimes \mathscr{O}_{\widehat{X}}(-2F_1 - 2F_2)) = 2K_X \otimes \mathfrak{m}_x^2$$

and by the Leray Spectral Sequence,

$$H^1((g_2 \circ g_1)_*(2M \otimes \mathscr{O}_{\widehat{X}}(-2F_1 - 2F_2))) = H^1(2M \otimes \mathscr{O}_{\widehat{X}}(-2F_1 - 2F_2)).$$

Local computation shows that

$$\mathscr{O}_{\overline{X}}(-2F) \cdot \mathscr{O}_{\widehat{X}} = \mathscr{O}_{\widehat{X}}(-F_1 - 2F_2) \text{ and that}$$
$$(\mathscr{O}_{\overline{X}}(-2F) \otimes m_{\overline{x}}) \cdot \mathscr{O}_{\widehat{X}} = \mathscr{O}_{\widehat{X}}(-2F_1 - 2F_2),$$

where $\mathfrak{m}_{\overline{x}}$ is the maximal ideal corresponding to $\overline{x}$.

Then Remarks c) and e) of Section 5 and Proposition 6.2 of [Lip69] and Lemma 4.2 show. that

$$g_*\mathscr{O}_{\widehat{X}}(-F_1 - 2F_2) = \mathscr{O}_{\overline{X}}(-2F) \text{ and}$$
$$g_*\mathscr{O}_{\widehat{X}}(-2F_1 - 2F_2) = \mathscr{O}_{\overline{X}}(-2F) \otimes \mathfrak{m}_{\overline{x}}$$

where $\mathfrak{m}_{\overline{x}}$ is the maximal ideal corresponding to $\overline{x}$. Thus we consider the exact sequence

$$0 \longrightarrow 2M \otimes \mathscr{O}_{\widehat{X}}(-2F_1 - 2F_2) \longrightarrow 2M \otimes \mathscr{O}_{\widehat{X}}(-F_1 - 2F_2) \longrightarrow 2M \otimes \mathscr{O}_{F_1}(-F_1 - 2F_2) \longrightarrow 0$$

and push it down to $\overline{X}$ to obtain

$$0 \longrightarrow g_*(2M \otimes \mathscr{O}_{\widehat{X}}(-2F_1 - 2F_2)) \longrightarrow 2L \otimes \mathscr{O}_{\overline{X}}(-2F) \longrightarrow \mathbf{k}_{\overline{x}} \longrightarrow 0, \tag{4.3.1}$$

where $\mathbf{k}_{\overline{x}}$ is the skycraper sheaf on $\overline{x}$ of dimension 1 obtained by restricting $2M \otimes \mathscr{O}_{F_1}(-F_1 - 2F_2)$ to $\overline{x}$. The above sequence is exact because $R^1 g_*(2M \otimes \mathscr{O}_{\widehat{X}}(-2F_1 - 2F_2))$ vanishes. Again by Leray Spectral Sequence, $H^1(2M \otimes \mathscr{O}_{\widehat{X}}(-2F_1 - 2F_2)) = H^1(g_*(2M \otimes \mathscr{O}_{\widehat{X}}(-2F_1 - 2F_2)))$, so we take cohomology in the above sequence (4.3.1). So, if we see that $H^1(2L \otimes \mathscr{O}_{\overline{X}}(-2F)) \neq 0$, then $H^1(2M \otimes \mathscr{O}_{\widehat{X}}(-2F_1 - 2F_2)) \neq 0$ and we are done. Then, to see that $H^1(2L \otimes \mathscr{O}_{\overline{X}}(-2F)) \neq 0$, we argue like this. From [GP07, Corollary 5.1], $K_{\overline{X}} = \overline{q}^* K_X + 2F$, hence

$$H^1(2L \otimes \mathscr{O}_{\overline{X}}(-2F)) = H^1(K_{\overline{X}} + L - 4F) = H^1(K_{\overline{X}}^* + p^*(2f)) = H^1(p^*(-2f))^*.$$

Now by [GP07, Corollary 4.3],

$$p_*\mathscr{O}_{\overline{X}} = \mathscr{O}_Y \oplus \mathscr{O}_Y(-2C_0 - 3f) \oplus \mathscr{O}_Y(-2C_0 - 3f) \oplus \mathscr{O}_Y(-3C_0 - 6f), \tag{4.3.2}$$

so

$$p_*p^*(\mathscr{O}_Y(-2f)) = \mathscr{O}_Y(-2f) \oplus \mathscr{O}_Y(-2C_0 - 5f) \oplus \mathscr{O}_Y(-2C_0 - 5f) \oplus \mathscr{O}_Y(-3C_0 - 8f) \tag{4.3.3}$$

Then, by (4.3.3) and the Leray Spectral Sequence, $h^1(p^*(-2f)) = h^1(\mathscr{O}_X(-2f)) = 1$.

Now we deal with Type 10. In this case $\overline{q}$ is the blow–up of X at x and a partial desingularization of X at x (recall that x is a D_4 singularity). Recall also that the exceptional divisor of $\overline{q}$ is a line F and in this case we have, as in Type 12, that $p^*C_0 = 4F$. The points of F are smooth except 3 points which are A_1 singularities and $F^2 = -1/2$. The local equation of X at x is $z^2t - t^3 - u^2 = 0$ and a local computation of the blowing up at x shows that $\mathfrak{m}_x\mathscr{O}_{\overline{X}} = \mathscr{O}_{\overline{X}}(-2F)$ and $\mathfrak{m}_x^2\mathscr{O}_{\overline{X}} = \mathscr{O}_{\overline{X}}(-4F)$. Now recall that we want to prove the nonvanishing of $H^1(2K_X \otimes \mathfrak{m}_x^2)$. Recall that $L = \overline{q}^* K_X = p^*(C_0 + 2f)$. By Lemma 4.2, $\overline{q}_*(\mathscr{O}_{\overline{X}}(-4F)) = \mathfrak{m}_x^2$, hence, by the projection formula and the Leray spectral sequence,

$$H^1(2K_X \otimes \mathfrak{m}_x^2) = H^1(p^*(C_0 + 4f)).$$

Recall also that in Type 10 (see [GP07, Corollary 4.3]),

$$p_*\mathscr{O}_{\overline{X}} = \mathscr{O}_Y \oplus \mathscr{O}_Y(-C_0 - 3f) \oplus \mathscr{O}_Y(-2C_0 - 3f) \oplus \mathscr{O}_Y(-3C_0 - 6f). \tag{4.3.4}$$

Thus, by the projection formula, the Leray spectral sequence and Serre duality, $h^1(p^*(C_0 + 4f)) = h^1(\mathscr{O}_Y(-2C_0 - 2f)) = h^1(\mathscr{O}_Y(-2f)) = 1$, so $H^1(2K_X \otimes \mathfrak{m}_x^2) \neq 0$, as wanted.

Now we deal with Type 9. In this case, $p^*C_0 = 2F$ and $F^2 = -2$, so x is an A_1 singularity, and $\overline{q}$ is the blow–up of X at x, which desingularizes X. Then $\mathfrak{m}_x^2\mathscr{O}_{\overline{X}} = \mathscr{O}_{\overline{X}}(-2F)$, so arguing as in the cases above and using Lemma 4.2, the projection formula and the Leray spectral sequence we have that $H^1(2L \otimes \mathscr{O}_{\overline{X}}(-2F)) = H^1(2K_X \otimes \mathfrak{m}_x^2)$. Recall that $L = \overline{q}^* K_X = p^*(C_0 + 2f)$, so $H^1(2L \otimes \mathscr{O}_{\overline{X}}(-2F)) = H^1(p^*(C_0 + 4f))$. Since $p_*\mathscr{O}_{\overline{X}}$ splits as (4.3.4) (see [GP07, Corollary 4.3]), we obtain that $h^1(\mathscr{O}_Y(-2f)) = 1$ like in Type 10.

Finally we study the case in which $\varphi^{-1}\{w\}$ consists of two points, x_1 and x_2. This is a quadruple Galois canonical cover of Type 11. We prove first that $|2K_X|$ does not separate x_1 and x_2. Recall (see [GP07, Corollary 5.1]) that x_1 and x_2 are smooth points and $\overline{q}$ is the blowing up of X at x_1 and x_2, so $\overline{q}_*\mathscr{O}_{\overline{X}}(-F_1 - F_2) = \mathfrak{m}_{x_1} \otimes \mathfrak{m}_{x_2}$, where F_1 and F_2 are the exceptional divisors of $\overline{q}$, which are -1–lines. Let f be a general fiber of the ruled surface Y and let $\overline{f}$ be the pullback to $\overline{X}$ of f by p. Then $\overline{f}$ is a smooth, connected curve of genus 4, meeting F_1 (respectively F_2) at one point $\overline{x}_1$ (respectively $\overline{x}_2$) transversally (see the proof of [GP07, Theorem 4.1]). Recall also that $\overline{q}^* K_X = L = p^*(C_0 + 2f)$ and that the morphism induced on $\overline{X}$ by $|2L|$ factors through φ_2. Then, if the restriction of $|2L|$ to $\overline{f}$ does not separate $\overline{x}_1$ and $\overline{x}_2$, then $\varphi_2(x_1) = \varphi_2(x_2)$. Note that the degree of $2L|_{\overline{f}}$ is 8. Recall also that $K_{\overline{X}} = p^*(C_0 + 2f) + F_1 + F_2$. Then, by adjunction formula,

$2L|_{\overline{f}}$ is the canonical of $\overline{f}$ plus the degree 2, effective divisor $(F_1+F_2)|_{\overline{f}} = \overline{x}_1 + \overline{x}_2$. Therefore the restriction of $|2L|$ to $\overline{f}$ does not separate $\overline{x}_1$ and $\overline{x}_2$.

Now we show that φ_2 is a local embedding at both x_1 and x_2. Let $Z' = \mathrm{Spec}(\mathscr{O}_{X,x_1}/\mathfrak{m}_{x_1}^2 \oplus \mathscr{O}_{X,x_2}/\mathfrak{m}_{x_2}^2)$. We will show that the cokernel of the homomorphism

$$H^0(2K_X) \longrightarrow H^0(2K_X|_{Z'}) \tag{4.3.5}$$

has dimension 1. Since $\varphi_2(x_1) = \varphi_2(x_2)$, in that case it would follow that φ_2 is a local embedding at both x_1 and x_2, so we would be done.

Thus, we show now that the cokernel of (4.3.5) has dimension 1. Since $H^1(2K_X) = 0$, this is equivalent to showing that $h^1(2K_X \otimes \mathfrak{m}_{x_1}^2 \otimes \mathfrak{m}_{x_2}^2) = 1$. For the latter, since $\overline{q}_* \mathscr{O}_{\overline{X}}(-2F_1 - 2F_2) = \mathfrak{m}_{x_1}^2 \otimes \mathfrak{m}_{x_2}^2$, from the projection formula and the Leray spectral sequence it follows

$$H^1(2K_X \otimes \mathfrak{m}_{x_1}^2 \otimes \mathfrak{m}_{x_2}^2) = H^1(2L \otimes \mathscr{O}_{\overline{X}}(-2F_1 - 2F_2)).$$

The latter cohomology group is equal to $H^1(p^*(C_0+4f))$. The splitting of $p_*\mathscr{O}_{\overline{X}}$ is (4.3.2) (see [GP07, Corollary 4.3]), so the projection formula, the Leray spectral sequence and Serre duality yield $h^1(p^*(C_0+4f)) = h^1(\mathscr{O}_Y(-2C_0-4f)) = h^1(\mathscr{O}_Y(-2f)) = 1$, so $h^1(2K_X \otimes \mathfrak{m}_{x_1}^2 \otimes \mathfrak{m}_{x_2}^2) = 1$. $\square$

5. Ring generators of the canonical ring of quadruple canonical covers

In this section we study the generators of the canonical ring of a quadruple Galois canonical cover X of minimal degree. Precisely we will find the degrees of the minimal generators of the canonical ring of X, and the number of generators in each degree. The first result of this section is a general result that gives a beautiful formula for the number of generators in degree 2 of the canonical ring of canonical covers, of arbitrary degree and irregularity, of surfaces of minimal degree. This result recovers part. of [GP03, Theorem 2.1], that gives the degrees and number of generators of the canonical ring of a regular surface S of general type with at worst canonical singularities, that is a canonical cover of a surface of minimal degree. One of the consequences of [GP03, Theorem 2.1] is that the number of extra generators needed in degree 2 depends only on the geometric genus $p_g(S)$ of S and on the degree n of Ψ; precisely [GP03, Theorem 2.1] tells that this number is $(n-2)(p_g(S)-2)$. This formula is generalized to canonical covers of arbitrary irregularity of surfaces of minimal degree in the following:

Theorem 5.1. *Let S be a surface of general type, normal with at worst canonical singularities and such that its canonical bundle is base–point–free. Let Ψ be the canonical morphism of S and let n be the degree of Ψ. Assume that the image of Ψ is a surface Y of minimal degree. Then, the degree 2 part of the canonical ring $R(S)$ of S is generated by the elements of degree 1 of $R(S)$ and by $(n-2)(p_g(S)-2)-q(S)$ linearly independent elements of degree 2.*

Proof. The part R_2 of degree 2 of $R(S)$ is $H^0(2K_S)$. Then the number of linearly independent elements of degree 2 which, in addition to the elements of degree 1, are needed to generate R_2 is the dimension of the cokernel of the multiplication map of global sections on S

$$H^0(K_S) \otimes H^0(K_S) \longrightarrow H^0(2K_S),$$

which is equal to the dimension of the cokernel of the multiplication map of global sections on Y

$$H^0(\Psi_* K_S) \otimes H^0(\Psi_* K_S) \xrightarrow{\gamma'} H^0(\Psi_* 2K_S).$$

We have an exact sequence

$$0 \longrightarrow \mathscr{O}_Y \xrightarrow{i} \Psi_* \mathscr{O}_S \longrightarrow \mathscr{F} \longrightarrow 0, \tag{5.1.1}$$

of sheaves on Y, where $\mathscr{F}$ is simply the cokernel of i. Let H be the hyperplane section of Y. Then $K_S = \Psi^* H$ and $H^0(K_S) = H^0(\Psi_* K_S) = H^0(H)$, last equality being induced by i. On the other hand,

$$H^0(H) \otimes H^0(H) \longrightarrow H^0(2H)$$

surjects, because Y is projectively normal. Then the image of γ' is $H^0(2H)$, which by (5.1.1) is a subspace of $H^0(\Psi_* 2K_S)$. Therefore the dimension of the cokernel of γ' is $h^0(\Psi_* 2K_S) - h^0(2H) = h^0(2K_S) - h^0(2H)$. By Riemann–Roch and the Kawamata–Viehweg vanishing theorem $h^0(2K_S) = K_S^2 + p_g(S) - q(S) + 1$. Let now C be a smooth curve in the linear system H, not meeting the singular locus of Y. Then $H^0(2H)$ fits in the following

$$0 \longrightarrow H^0(H) \longrightarrow H^0(2H) \longrightarrow H^0(2H|_C) \longrightarrow 0$$

exact sequence, since $H^1(H) = 0$, which follows from the fact that Y is a regular surface. Since $C = \mathbf{P}^1$, $h^0(2H|_C) = 2H^2 + 1$, so $h^0(2H) = p_g(S) + 2H^2 + 1$. Now we have that $K_S^2 = nH^2$ and, since Y is a surface of minimal degree, $H^2 = p_g(S) - 2$. All this yields $h^0(2K_S) - h^0(2H) = (n-2)(p_g(S) - 2) - q(S)$ □

The result [GP03, Theorem 2.1] tells among other things that, unless $n = 2$, the canonical ring of S is generated in degree less than or equal to 3. Theorem 5.1 gives a nice, uniform formula for the number of generators in degree 2, depending only on the geometric and arithmetic genus of S. Thus a natural question to ask is whether there exists a uniform formula or pattern for the number of generators in degree 3, similar to the one for generators of degree 2. Indeed, [GP03, Theorem 2.1] gives us such a formula when S is regular (precisely, the number of extra generators in degree 3 is $p_g(S) - 3$ if S is regular and $n \neq 2$). However, irregular quadruple Galois canonical covers of Types 5.1, 6.1, 7, 8 show that this formula cannot be generalized for an arbitrary S, as one can see in Theorem 5.2 below:

Theorem 5.2. *Let $\varphi : X \longrightarrow W$ be a quadruple Galois canonical cover of a surface of minimal degree. Then the canonical ring of X is generated in degree less than or equal to* 3. *Precisely, the canonical ring of X is generated by its part of degree* 1, *by $2p_g(X) - 4 - q(X)$ extra generators in degree* 2, *and by $\delta(X)$ extra generators in degree* 3, *where*

a) $\delta(X) = p_g(X) - 3$ *if X is of Type 1, 2, 3, 4, 5.2, 6.2, 9, 10, 11 or 12.*
b) $\delta(X) = 4$, *if X is of Type 5.1 or 6.1;*
c) $\delta(X) = 5m - 1$ *if X is of Type 7;*
d) $\delta(X) = 9$, *if X is of Type 8.1;*
e) $\delta(X) = 5m$ *if X is of Type 8.2.*

Proof. Let R be the canonical ring of X and $R_n = H^0(nK_X)$ its part of degree n. If X is regular (i.e., if X is of Type 1, 2, 3, 4, 9, 10, 11 or 12) the result follows from [GP03, Theorem 2.1]. The part of the result regarding R_2, for any surface X, follows from Theorem 5.1.

Then it only remains to prove that R is generated in degree less than or equal to 3 and to find the number of extra generators of degree 3, if X is of Type 5, 6, 7 or 8. For this we will study the multiplication maps of global sections of line bundles on X

$$H^0(K_X) \otimes H^0(nK_X) \xrightarrow{\gamma_n} H^0((n+1)K_X),$$

when $n \geq 2$. The dimension of the cokernel of γ_2 is the number of linearly independent elements of degree 3 which are required, together with the elements of R_1 and R_2, to generate R_3. On the other hand, if γ_n is surjective for all $n \geq 3$, then R is generated by its elements of degree less than or equal to 3. Now, to study the maps γ_n, we look at the $\mathscr{O}_W$–algebra structure of $\varphi_* \mathscr{O}_X$ and use Lemma 2.1 and arguments similar to those used in Section 2. Recall that by Theorem 1.3, $W = \mathbf{F}_0$

and that the splitting (2.0.2) of $\varphi_*\mathcal{O}_X$ is

$$(5.2.1)\quad \begin{array}{ll} \text{if } X \text{ is of Type 5,} & \mathcal{O}_W \oplus \mathcal{O}_W(-C_0-(m+2)f) \oplus \mathcal{O}_W(-2C_0) \oplus \mathcal{O}_W(-3C_0-(m+2)f); \\ \text{if } X \text{ is of Type 6,} & \mathcal{O}_W \oplus \mathcal{O}_W(-C_0-(m+2)f) \oplus \mathcal{O}_W(-2C_0) \oplus \mathcal{O}_W(-3C_0-(m+2)f); \\ \text{if } X \text{ is of Type 7,} & \mathcal{O}_W \oplus \mathcal{O}_W(-(m+1)f) \oplus \mathcal{O}_W(-3C_0-f) \oplus \mathcal{O}_W(-3C_0-(m+2)f) \\ \text{if } X \text{ is of Type 8,} & \mathcal{O}_W \oplus \mathcal{O}_W(-(m+2)f) \oplus \mathcal{O}_W(-3C_0) \oplus \mathcal{O}_W(-3C_0-(m+2)f). \end{array}$$

We look first at γ_2. Recall that $H^0(\varphi_*K_X)$ has in all cases one nonzero summand, namely, $H^0(C_0+mf)$. First we consider X of Type 5.1 or 6.1. It follows from 5.2.1 that the splitting (2.1.1) of $H^0(\varphi_*2K_X)$ has two nonzero summands, namely $H^0(2C_0+2f)$ and $H^0(2f)$, corresponding to $\mathcal{O}_W$ and L_2^* of 2.0.2. On the other hand, the splitting (2.1.1) of $H^0(\varphi_*3K_X)$ has four non zero summands, namely, $H^0(3C_0+3f), H^0(2C_0), H^0(C_0+3f)$ and $H^0(\mathcal{O}_W)$. Then Lemma 2.1 implies that γ_2 is not surjective. More precisely, since γ_2 splits in several summands according to the algebra structure of $\varphi_*\mathcal{O}_X$, described in [GP08, Remark 3.1], it follows that the image of γ_2 is contained in $H^0(3C_0+3f) \oplus H^0(C_0+3f)$. Moreover, γ_2 surjects onto $H^0(3C_0+3f) \oplus H^0(C_0+3f)$ if the following multiplication maps on W

$$H^0(C_0+f) \otimes H^0(2C_0+2f) \longrightarrow H^0(3C_0+3f)$$
$$H^0(C_0+f) \otimes H^0(2f) \longrightarrow H^0(C_0+3f)$$

surject. This follows from Lemma 2.4, e). Then the dimension of the cokernel of γ_2 is $h^0(2C_0)+h^0(\mathcal{O}_W)=4$.

Arguing similarly we see that in all other Types 5.2, 6.2, 7 and 8, we note that, from 5.2.1, the splitting (2.1.1) of $H^0(\varphi_*2K_X)$ has these nonzero summands:

$H^0(2C_0+2mf) \oplus H^0(C_0+(m-2)f) \oplus H^0(2mf)$, if X is of Type 5.2 or 6.2 (recall that in such a case, $m \geq 2$);
$H^0(2C_0+2mf) \oplus H^0(2C_0+(m-1)f)$, if X is of Type 7;
$H^0(2C_0+2f)$, if X is of Type 8.1; and
$H^0(2C_0+2mf) \oplus H^0(2C_0+(m-2)f)$, if X is of Type 8.2 (recall that in such a case, $m \geq 2$).

On the other hand, since γ_2 splits in several summands according to the algebra structure of $\varphi_*\mathcal{O}_X$, described in [GP08, Remark 3.1], it follows that the image of γ_2 is contained in the subspace B of $H^0(\varphi_*3K_X)$, where B is

$H^0(3C_0+3mf) \oplus H^0(2C_0+(2m-2)f) \oplus H^0(C_0+3mf)$, if X is of Type 5.2 or 6.2
$H^0(3C_0+3mf) \oplus H^0(3C_0+(2m-1)f)$, if X is of Type 7;
$H^0(3C_0+3f)$, if X is of Type 8.1; and
$H^0(3C_0+3mf) \oplus H^0(3C_0+(2m-2)f)$, if X is of Type 8.2

By Lemma 2.4, e), γ_2 in fact surjects onto B so the cokernel of γ_2 is

$H^0((2m-2)f)$, if X is of Type 5.2 or 6.2;
$H^0((3m-1)f) \oplus H^0((2m-2)f)$, if X is of Type 7;
$H^0(3C_0) \oplus H^0(3f) \oplus H^0(\mathcal{O}_W)$, if X is of Type 8.1; and
$H^0(3mf) \oplus H^0((2m-2)f$, if X is of Type 8.2.

Then the dimension of the cokernel of γ_2 is

$2m-1$ if X is of Type 5.2 or 6.2;
$5m-1$ if X is of Type 7;
9 if X is of Type 8.1; and
$5m$ if X is of Type 8.2.

Recall that $2m$ is the degree of W, and W is a surface of minimal degree, so $2m = p_g(X)-2$; therefore the number of extra generators of R in degree 3 is $p_g(X)-3$ if X is of Type 5.2 or 6.2.

Finally from 5.2.1, the splitting (2.1.1) of $H^0(\varphi_* nK_X)$ has four nonzero summands if $n \geq 3$. Then the splitting of γ_n in summands according to the algebra structure of $\varphi_*\mathscr{O}_X$ and Lemma 2.4, e) imply the surjectivity of γ_n for all $n \geq 3$.

□

Remark 5.3. It is a common phenomenon that, if a graded ring R is generated in degree less than or equal to 2, then the Veronese 2–subring R' of R is generated in degree 1. Theorems 2.3, 2.5, 2.6 and 5.2 show the existence of several families of surfaces X of general type (both regular and irregular; precisely, surfaces X of Types 1, 2, 3, 4, 5.2 and 6.2) such that the Veronese 2–subring of their canonical ring is generated in degree 1, despite the fact that their canonical ring is not generated in degree less than or equal to 2.

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Departamento de Álgebra, Universidad Complutense de Madrid
E-mail address: gallego@mat.ucm.es

Department of Mathematics, University of Kansas
E-mail address: purna@math.ku.edu